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Degree of L_1 Approximation to Integrable Functions by Modified Bernstein Polynomials

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1. INTRODUCTION AND RESULTS

If f is a function defined on [0, 1], the Bernstein polynomial $B_n(f)$ of f is

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n) p_{n,k}(x),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}.$$

S. Bernstein [1] has proved that for every continuous function f on [0, 1],

$$\max_{0 \leqslant x \leqslant 1} | B_n(f, x) - f(x)| \to 0 \qquad (n \to \infty).$$

A more precise version of this result due to T. Popoviciu [2] states that

$$\max_{0 \leqslant x \leqslant 1} | B_n(f, x) - f(x)| \leqslant \tfrac{5}{4} \omega_f(n^{-1/2})$$

where ω_f is the uniform modulus of continuity of f defined by

$$\omega_f(h) = \max\{| f(x) - f(y)| : x, y \in [0, 1], | x - y | \leqslant h\}.$$

A small modification of Bernstein polynomials due to L. A. Kantorovič [3]

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makes it possible to approximate Lebesgue integrable functions in the L_1 norm by the modified polynomials

$$P_n(f, x) = (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right) p_{n,k}(x).$$

The L_1 analog of Bernstein's result was established by G. G. Lorentz [4] who has proved that for every Lebesgue integrable function f on $[0, 1]$,

$$\int_0^1 | P_n(f, x) - f(x)|\, dx \to 0 \ (n \to \infty).$$

As far as estimates of the degree of approximation to Lebesgue integrable functions by the polynomials $P_n(f)$ in the L_1 norm are concerned, very little is known. A result which gives the degree of approximation to f by $P_n(f)$ for a very special class of Lebesgue integrable functions f is due to W. Hoeffding [5]. Hoeffding's result may be stated as follows.

If f is a Lebesgue integrable function on $[0, 1]$, of bounded variation on every closed subinterval of $(0, 1)$, then

$$\int_0^1 | P_n(f, x) - f(x)|\, dx \leqslant (2/e)^{1/2} J(f) n^{-1/2},$$

where

$$J(f) = \int_0^1 x^{1/2}(1-x)^{1/2}\, | df(x)|.$$

This result is useful when $J(f) < \infty$.

In this paper we shall show that

$$\int_0^1 x^{1/2}(1-x)^{1/2}\, | P_n(f, x) - f(x)|\, dx$$

can be estimated in terms of the L_1 modulus of continuity

$$\omega_f(h)_{L_1} = \sup_{|t| \leqslant h} \int_0^1 | f(x+t) - f(x)|\, dx.$$

We assume here and in the rest of the paper that the function f is extended to $(-\infty, \infty)$ by periodicity with period 1 (its value at the integers is immaterial). The L_1 norm with the weight function $w(x) = x^{1/2}(1-x)^{1/2}$ seems to be a more convenient norm than the usual L_1 norm for the study of approximation properties of modified Bernstein polynomials.

Our result may be stated as follows.

THEOREM. *Let f be a Lebesgue integrable on function* [0, 1]. *Then, for* $n \geqslant 2$,

$$\int_0^1 x^{1/2}(1-x)^{1/2} \mid P_n(f, x) - f(x) \mid dx \leqslant \frac{2\pi^2}{3} \omega_f(n^{-1/2})_{L_1}.$$

2. LEMMAS

The proof of our theorem is based on two lemmas.

LEMMA 1. *If f is a Lebesgue integrable function on* [0, 1], *then, for* $n \geqslant 2$ *and for* $x \in (0, 1)$, *we have*

$$x(1-x)(P_{n-1}(f, x) - f(x)) = \sum_{k=0}^{n} np_{n,k}(x)\left(\frac{k}{n} - x\right) \int_0^{k/n-x} (f(x+t) - f(x))\, dt.$$

Proof. We have

$$P_{n-1}(f, x) = \int_0^1 K_n(x, t) f(t)\, dt,$$

where

$$K_n(x, t) = n \sum_{k=0}^{n-1} p_{n-1,k}(x) \chi_{(k/n,(k+1)/n]}(t),$$

$\chi_{(k/n,(k+1)/n]}(t)$ being the characteristic function of $(k/n, (k+1)/n]$. By partial summation we find that

$$\sum_{k=0}^{n-1} p_{n-1,k}(x)\, \chi_{(k/n,(k+1)/n]}(t) = p_{n-1,n-1}(x)\, \chi_{[0,1]}(t) - p_{n-1,0}(x)\, \chi_{[0,0]}(t) + \sum_{k=1}^{n-1} (p_{n-1,k-1}(x) - p_{n-1,k}(x))\, \chi_{[0,k/n]}(t).$$

Since

$$p_{n-1,k-1}(x) - p_{n-1,k}(x) = \left(\binom{n-1}{k-1}(1-x) - \binom{n-1}{k}(x)\right) x^{k-1}(1-x)^{n-k-1} = \binom{n}{k}\left(\frac{k}{n} - x\right) x^{k-1}(1-x)^{n-k-1},$$

we have

$$x(1-x)(p_{n-1,k-1}(x) - p_{n-1,k}(x)) = (k/n - x)\, p_{n,k}(x),$$

and it follows that

$$x(1-x)\,K_n(x,t)=\sum_{k=0}^{n} np_{n,k}(x)\left(\frac{k}{n}-x\right)\chi_{[0,k/n]}(t).$$

Hence,

$$x(1-x)\,P_{n-1}(f,x)=\sum_{k=0}^{n} np_{n,k}(x)\left(\frac{k}{n}-x\right)\int_0^{k/n} f(t)\,dt$$

$$=\sum_{k=0}^{n} np_{n,k}(x)\left(\frac{k}{n}-x\right)\int_x^{k/n} f(t)\,dt$$

and the proof of the lemma is complete, since

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 p_{n,k}(x)=x(1-x)/n.$$

Our second lemma is a more precise version of a known inequality (see [6, p. 15]).

LEMMA 2. *For $n\geqslant 2$ and $x\in[0,1]$ we have*

$$\sum_{k=0}^{n}|k/n-x|^5\,p_{n,k}(x)\leqslant x(1-x)/n^{5/2}.$$

Proof. We have

$$\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^5 p_{n,k}(x)\leqslant\left(\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^4 p_{n,k}(x)\right)^{1/2}\left(\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^6 p_{n,k}(x)\right)^{1/2}$$

and the result follows, since

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^4 p_{n,k}(x)=\frac{x(1-x)}{n^2}\left(3x(1-x)+\frac{1-6x(1-x)}{n}\right)$$

$$\leqslant\frac{x(1-x)}{n^2}$$

and

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^6 p_{n,k}(x)$$

$$=\frac{x(1-x)}{n^3}\left(15x^2(1-x)^2+\frac{25x(1-x)-130x^2(1-x)^2}{n}\right.$$

$$\left.+\frac{1-6x(1-x)-36x^2(1-x)^2+168x^3(1-x)^3}{n^2}\right)$$

$$\leqslant\frac{x(1-x)}{n^3}$$

for $x\in[0,1]$.

3. Proof of the Theorem

Let $x \in (0, 1)$. By Lemma 1 we have

$$
\begin{aligned}
x(1-x) | P_{n-1}(f, x) - f(x) | &\leq \sum_{k=0}^{n} np_{n,k}(x) \left| \frac{k}{n} - x \right| \left| \int_0^{k/n-x} (f(x+t) - f(x))\, dt \right| \\
&\leq \sum_{k=0}^{n} np_{n,k}(x) \left| \frac{k}{n} - x \right| \int_{-|k/n-x|}^{|k/n-x|} | f(x+t) - f(x) |\, dt \\
&\leq \sum_{r=0}^{[1/\delta]} I_{n,r}(x),
\end{aligned}
$$

where $\delta \in (0, 1)$ and

$$I_{n,r}(x) = \sum_{r\delta < |k/n-x| \leq (r+1)\delta} np_{n,k}(x) \left| \frac{k}{n} - x \right| \int_{-|k/n-x|}^{|k/n-x|} | f(x+t) - f(x) |\, dt.$$

Clearly

$$I_{n,r}(x) \leq S_r(n, \delta; x) \int_{-(r+1)\delta}^{(r+1)\delta} | f(x+t) - f(x) |\, dt,$$

where

$$S_r(n, \delta; x) = \sum_{r\delta < |k/n-x| \leq (r+1)\delta} np_{n,k}(x) \left| \frac{k}{n} - x \right|.$$

Hence, it follows that

$$x(1-x) | P_{n-1}(f, x) - f(x) | \leq \sum_{r=0}^{[1/\delta]} S_r(n, \delta; x) \int_{-(r+1)\delta}^{(r+1)\delta} | f(x+t) - f(x) |\, dt. \tag{1}$$

Next we shall estimate the coefficients $S_r(n, \delta; x)$ for $r = 0$ and $1 \leq r \leq [1/\delta]$. We have first

$$
\begin{aligned}
S_0(n, \delta; x) = \sum_{|k/n-x| \leq \delta} np_{n,k}(x) \, | k/n - x | &\leq \sum_{k=0}^{n} np_{n,k}(x) \, | k/n - x | \\
&\leq n^{1/2} x^{1/2} (1-x)^{1/2}.
\end{aligned} \tag{2}
$$

Next, for $1 \leq r \leq [1/\delta]$, we have, by Lemma 2,

$$
\begin{aligned}
S_r(n, \delta; x) &\leq n(r+1)^{-4} \delta^{-4} \sum_{r\delta < |k/n-x| \leq (r+1)\delta} | k/n - x |^5 \, p_{n,k}(x) \\
&\leq n(r+1)^{-4} \delta^{-4} \sum_{k=0}^{n} | k/n - x |^5 \, p_{n,k}(x) \\
&\leq n^{-3/2} x(1-x)(r+1)^{-4} \delta^{-4},
\end{aligned}
$$

From (1), (2) and (3) it follows that

$$x^{1/2}(1-x)^{1/2}\,|\,P_{n-1}(f,x)-f(x)|$$
$$\leqslant n^{1/2}\int_0^1 |f(x+t)-f(x)|\,dt$$
$$+\tfrac{1}{2}n^{-3/2}\,\delta^{-4}\sum_{r=1}^{[1/\delta]}(r+1)^{-4}\int_{-(r+1)\delta}^{(r+1)\delta}|f(x+t)-f(x)|\,dt.$$

Integrating this inequality and taking into account that

$$\int_{-(r+1)\delta}^{(r+1)\delta}\left(\int_0^1 |f(x+t)-f(x)|\,dx\right)dt \leqslant 2(r+1)\,\delta\omega_f((r+1)\delta)_{L_1},$$

we find that

$$\int_0^1 x^{1/2}(1-x)^{1/2}\,|\,P_{n-1}(f,x)-f(x)|\,dx$$
$$\leqslant 2n^{1/2}\,\delta\omega_f(\delta)_{L_1}+n^{-3/2}\,\delta^{-3}\sum_{r=1}^{[1/\delta]}(r+1)^{-3}\,\omega_f((r+1)\delta)_{L_1}.$$

Choosing here $\delta = n^{-1/2}$, we find that

$$\int_0^1 x^{1/2}(1-x)^{1/2}\,|\,P_{n-1}(f,x)-f(x)|\,dx$$
$$\leqslant 2\omega_f(n^{-1/2})_{L_1}+\sum_{r=1}^{[n^{1/2}]}(r+1)^{-3}\,\omega_f((r+1)/n^{1/2})_{L_1}$$
$$\leqslant 2\sum_{k=1}^{[n^{1/2}]+1}k^{-3}\,\omega_f(k/n^{1/2})_{L_1}.$$

Since the L_1 modulus of continuity is a subadditive function, we have, for every $0 < h_1 \leqslant h_2$,

$$2\,\frac{\omega_f(h_1)_{L1}}{h_1} \geqslant \frac{\omega_f(h_2)_{L1}}{h_2}$$

(see [7], p. 112). In particular we have, for $k \geqslant 1$,

$$\omega_f(k/n^{1/2})_{L_1} \leqslant 2k\omega_f(n^{-1/2})_{L_1}.$$

Hence,

$$\int_0^1 x^{1/2}(1-x)^{1/2}\,|\,P_{n-1}(f,x)-f(x)|\,dx \leqslant 4\omega_f(n^{-1/2})_{L_1}\sum_{k=1}^{\infty}k^{-2}$$
$$\leqslant \frac{2\pi^2}{3}\,\omega_f(n^{-1/2})_{L_1}$$

and the theorem is proved.

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